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Dispersion and Shock-Wave Structure*

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1. INTRODUCTION

We consider a system of conservation laws in a single space variable of the form

$$u_t + F(u)_x = \varepsilon A u_{xx} + \varepsilon^2 B u_{xxx}. \tag{1.1}$$

Here F is a smooth function of $u \in \mathbb{R}^n$, A and B are $n \times n$ positive definite symmetric matrices, and ε is a non-negative constant. The quantities Au_{xx} and Bu_{xxx} correspond, respectively, to dissipation and dispersion terms. We assume that when $\varepsilon = 0$, the resulting system is hyperbolic, and genuinely non-linear in the sense of Lax [11]. Under these conditions, it is well-known that the "reduced" system admits shock-wave solutions. For $\varepsilon > 0$, system (1.1) admits progressive-wave solutions, and we consider the problem of finding conditions on A and B which will guarantee that the shock waves can be obtained as limits of progressive waves as $\varepsilon \to 0$. When this is the case, we say that the pair (A, B) is "admissible" (for the particular shock wave). The admissibility problem is connected with finite difference approximations to (1.1) [12], as well as to the so-called "viscosity" method for the reduced problem [17].

In this paper, we find conditions which imply that the pair (A, B) is admissible. Thus, for general n, we show that if A and B are "near" scalar matrices, and $A > 0$, then (A, B) is admissible for all sufficiently weak shocks.[1] In the case $n = 2$, we consider shocks of arbitrary strength, and we obtain admissibility criteria. These criteria say, roughly, that A "dominates"

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[1] This restriction is necessary since for these general systems, one only knows the existence of weak shocks; see [11].

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B. Thus, for example, in the case where A is a positive definite matrix, $A = \text{diag}\,(a, d)$ and $B = \text{diag}\,(b_1, b_2)$, our conditions imply that the quantity

$$\min(a, d) \min(a/|b_1|, d/|b_2|)$$

is bounded away from zero by a constant depending only on F and the given shock.

In order to put our results into perspective we shall briefly review some of the literature on this subject. In a series of paper [2–8], the admissibility question has been discussed under the assumption that either A or B is zero. Thus, for arbitrary n, it was shown in [6] that $(A, 0)$ is admissible for sufficiently weak shocks provided that A is sufficiently close to the identity. When A is no longer near to the identity, the result is fase; indeed, in [3] it is shown that there is a positive-definite symmetric matrix A which is inadmissible for all shocks. In addition, if F is a gradient, $(0, B)$ is generally not admissible for all sufficiently weak shocks; see [5]. In the case $n = 2$ (for the systems studied in [16]), it was shown that $(A, 0)$, with A positive diagonal, is admissible for all shocks, while for some shocks, $(A, 0)$ will be admissible for all positive definite symmetric A. Some of these results were subsequently obtained in [9] by different methods; see also the related works [1, 10, 13].

As discussed in [3], the admissibility question reduces to a problem in ordinary differential equations, namely, that of finding traveling-wave solutions of (1.1). Such solutions satisfy an ordinary differential equation in $\mathbb{R}^{2n}$, and the problem is to find a solution curve which "connects" two rest points. Our technique is to employ Conley's notion of an isolated invariant set and its index (see [2]). In Section 2 we give a very brief discussion of this, together with the relevant background from the theory of shock waves.

The admissibility criteria, based on Conley's index, require that the flow be gradient-like in an open set contaning the rest points. In our cases, we cannot construct the "global" gradient function, and the admissibility technique requires slight modification. In Section 2 we give one such modification, which is actually implicit in [6]. This is used in Section 3, where we consider the case of weak shocks (n arbitrary). In Section 4 we take $n = 2$ and consider shocks of arbitrary strength. We let A be symmetric positive definite and B diagonal. We find conditions which allow us to construct an isolating neighborhood containing both rest points. We show that the flow is gradient-like in this neighborhood, outside of two small isolating blocks about the rest points. To compute the index, we use the invariance of the index under continuation and first deform A and B to scalar matrices, and then deform the equations to those with arbitrarily weak shocks. Then using computations obtained in Section 3, we find that the index is trivial. This implies that (A, B) is admissible. In Section 5 we consider an example of a gradient F in $\mathbb{R}^2$, for which all pairs (A, B) are admissible, where $A > 0$ and B is symmetric and invertible (for shocks of a given characteristic family). The main interest in this example is in the method of the proof.

The admissibility theorems in this paper first appeared in the unpublished work [14], with some different proofs. Indeed, the proofs in Section 4 are much simpler than the original ones; in particular, we replace a difficult topological construction by the continuation argument outlined above. We also extend the results in [14], and give new examples.

2. Formulation of the Problem

Consider the hyperbolic system of conservation laws

$$u_t + F(u)_x = 0, \tag{2.1}$$

where $u \in \mathbb{R}^n$, $x \in \mathbb{R}$, $t \geqslant 0$ and F is a smooth mapping from some open subset of $\mathbb{R}^n$ into $\mathbb{R}^n$. The hyperbolicity assumption means that the Jacobian matrix dF has real and distinct eigenvalues $\lambda_1(u) < \cdots < \lambda_n(u)$. The corresponding left and right eigenvectors will be denoted by l_i and r_i, respectively, $i = 1,..., n$. Following Lax [11], we require that system (2.1) be genuinely nonlinear; i.e., $l_i\, d^2F(r_i, r_i) \neq 0$, $i = 1,..., n$, where d^2F is a bilinear form, the second derivative of the mapping F. We normalize l_i and r_i by choosing

$$l_i\, d^2F(r_i, r_i) = 1 \qquad \text{and} \qquad l_i r_i = 1, \tag{2.2}$$

for each i.

By a k-shock wave solution of (2.1) is meant a solution of the form

$$\begin{aligned} u(x, t) &= u_l, \qquad x < st, \\ &= u_r, \qquad x > st, \end{aligned} \tag{2.3}$$

where u_l and u_r are constant vectors, and s is the shock speed. This solution is also required to satisfy both the "jump" condition

$$s(u_l - u_r) = F(u_l) - F(u_r), \tag{2.4}$$

and the "entropy" conditions

$$\lambda_k(u_r) < s < \lambda_{k+1}(u_r), \qquad \lambda_{k-1}(u_l) < s < \lambda_k(u_l). \tag{2.5}$$

We now consider systems of the form

$$u_t + F(u)_x = \varepsilon A u_{xx} + \varepsilon^2 B u_{xxx}, \tag{2.6}$$

obtained from (2.1) by the addition of (formally small) dissipative terms $\varepsilon A u_{xx}$, and dispersive terms $\varepsilon^2 B u_{xxx}$. Here A and B are both taken to be positive definite symmetric matrices. A progressive wave solution of (2.6) is one of the form $u = u(\xi)$, $\xi = (x - st)/\varepsilon$. Such a solution satisfies the ordinary differential equation

$$Bu'' + Au'' = (F(u) - su)', \qquad ' = d/d\xi.$$

We integrate this equation and obtain

$$Bu'' + Au' = F(u) - su + D, \tag{2.7}$$

where D is a constant. Using (2.4), we see that if $D = -F(u_l) + su_l$, then the right-hand side of (2.7) vanishes at both u_l and u_r. In the remainder of this paper, we restrict attention to the case where $u_l = F(u_l) = 0$; this is no loss in generality. Thus $D = 0$, and (2.7) can be written as the first order system

$$\begin{aligned} Bu' &= w, \\ w' &= F(u) - su - ACw, \qquad C = B^{-1}. \end{aligned} \tag{2.8}$$

Observe that as $\varepsilon \to 0$, $\xi \to \pm\infty$, depending on the sign of $x - st$. It follows that (A, B) is admissible, for the given shock (2.3), provided that there is an orbit of (2.8) satisfying[2]

$$\lim_{\xi \to -\infty} (u(\xi), w(\xi)) = (0, 0), \qquad \lim_{\xi \to +\infty} (u(\xi), w(\xi)) = (u_r, 0). \tag{2.9}$$

Thus, our problem is to find an orbit of (2.8) which satisfies the boundary conditions (2.9).

In order to solve this problem, we use Conley's notion of the index of an isolated invariant set, and we now very briefly outline the relevant parts of the theory; see [2] for a complete development.

Given an autonomous equation on $\mathbb{R}^n$, we will denote the associated flow by $x \cdot t$. An isolating neighborhood for the flow is a bounded open set $U \subset \mathbb{R}^n$ such that if $p \in \partial U$, then $p \cdot \mathbb{R} \not\subset \bar{U}$. The maximal invariant set, $S = S(U)$, in U is called an isolated invariant set. An isolating block B is an isolating neighborhood with the property that each boundary point of B is a strict-exit, or strict-entrance, point under the flow. Isolated invariant sets are always realized as the maximal invariant set in some isolating neightborhood. If b^+ (resp. b^-) denotes the strict-exit (resp. entrance) points on ∂B, then if B/b^+ (or B/b^-) is not homotopic to a point, the maximal invariant set S in B is non-empty. The homotopy type of B/b^+ depends only on S and is called the Conley index of S; we write it as $h(S)$. If S is a hyper-

[2] In [5] this sufficient condition is also shown to be necessary.

bolic rest point (i.e., the matrix of the linearization about S has no eigenvalue with zero real part), then S is an isolated invariant set and $h(S)$ is the pointed k-sphere, where k is the dimension of the unstable manifold of S.

The Conley index satisfies a useful continuation property which we now describe. Observe first that if N is an isolating neighborhood for a vector field V, it is also one for all nearby vector fields; the corresponding isolated invariant sets are called continuations of S, and we say that they are related to S by continuation. If this relation is defined to be transitive, then the continuations have the same Conley index as S. This often gives one a way of computing indices.

In [6], the following theorem was used to prove admissibility results.

THEOREM 2.1. *Let V be a vector field in $\mathbb{R}^n$ which admits two rest points x_0 and x_1, not both of which are degenerate. Suppose that x_0 and x_1 are the only rest points contained in an isolating neighborhood N, and*[3] *$h(S(N)) \neq h(x_0) \vee h(x_1)$. Then there is an orbit of V in N which is different from x_0 and x_1. If V is also gradient-like in N, then this orbit connects x_0 to x_1.*

We shall not be able to use this theorem directly since we cannot show that our equations are gradient-like in entire isolating neighborhoods. Thus we must rely on a somewhat different strategy (albeit in the same spirit as above). Namely, we have the following theorem (see [6]).

THEOREM 2.2. *Let B be an isolating block for a vector field V. Assume that there is a hyperplane Γ separating B into two blocks B_1 and B_2, and that $\Gamma = b_1^- \cap b_2^+$. If $h(S(B)) \neq h(S(B_1)) \vee h(S(B_2))$, then there is an orbit of V whose ω-limit set is in B_1 and whose α-limit set is in B_2. If, in addition, each B_i contains precisely one rest point p_i, and if V is gradient-like in each B_i, then this orbit runs from p_1 to p_2.*

The hypothesis on the indices implies that there is an orbit of V in B which is not contained in either B_i. Since $B_1 \cap B_2 = b_1^- \cap b_2^+$, this orbit crosses from B_2 to B_1 precisely once and has its α- and ω-limit sets in the required B_i's. The gradient-like nature of V forces the orbit to connect the two rest points.

3. Weak Shocks in General Systems

We consider general hyperbolic systems of conservation laws in n unknowns $u = (u_1, u_2, \dots, u_n)$ of the form (2.1).

[3] The symbol $\vee$ denotes wedge-product; see [2].

We let $\lambda_1(u) < \cdots < \lambda_n(u)$ denote the eigenvalues of dF, with corresponding left and right eigenvectors l_i and r_i, respectively, $i = 1, 2,..., n$. We assume that (2.1) is genuinely nonlinear, and that the eigenvectors are normalized according to (2.2).

In order to construct isolating neighborhoods, we recall from [11] certain estimates on weak shocks. Thus, for $\rho < 0$, and ρ sufficiently close to 0, we can write the totality of states connected to $u = 0$ by a k-shock (the "k-shock curve"), in the form

$$V_k(\rho) = \rho r_k + O_2(\rho).$$

The corresponding shock speed can be written as

$$s(\rho) = \lambda_k + \rho/2 + O_2(\rho). \tag{3.2}$$

Furthermore, there is a ball around $u = 0$ such that for ρ sufficiently small, O and $V_k(\rho)$ are the only critical points of the vector field $F(u) - s(\rho)\, u$ in this ball.

We now consider Eq. (1.1), where A and B are scalar matrices, $A = aI$, $B = bI$, $a > 0$. The corresponding equations (2.8) become

$$\begin{aligned} u' &= cw, \\ w' &= F(u) - su - acw, \qquad c = b^{-1}. \end{aligned} \tag{3.3}$$

If $E(u, w) = (cw, F(u) - su - acw)$, then

$$dE = \begin{bmatrix} 0 & c \\ dF - s & -ac \end{bmatrix}. \tag{3.4}$$

We see that dE has eigenvalues μ satisfying $\mu(\mu + ac) - v_i c = 0$, where $v_i = \lambda_i - s$, $i = 1, 2,..., n$. Thus

$$\begin{aligned} \mu_i &= (-a - \sqrt{a^2 + 4v_i b})/2b, \qquad i = 1, 2,..., n, \\ \mu_{i+n} &= (-a + \sqrt{a^2 + 4v_i b})/2b, \qquad i = 1, 2,..., n. \end{aligned} \tag{3.5}$$

If $a^2 + 4v_i b > 0$, $i = 1, 2,..., n$, then dE admits a complete set of corresponding left (L_i), and right (R_i) eigenvectors, defined by

$$\begin{aligned} L_i &= (-b\mu_{i+n} l_i, l_i), \qquad L_{i+n} = (-b\mu_i l_i, l_i), \\ R_i &= (r_i, b\mu_i r_i)^t, \qquad R_{i+n} = (r_i, b\mu_{i+n} r_i)^t, \quad i = 1,.... \end{aligned} \tag{3.6}$$

If $a^2 + 4bv_j < 0$ (where $j < k$ by (2.5)), then $R_j = (r_j, -\frac{1}{2}ar_j)^t$, and $R_{j+n} = (0, \frac{1}{2}\sqrt{|a^2 + 4bv_j|}\, r_j)^t$, with similar changes for L_j and L_{j+n}. If $a^2 + 4bv_j = 0$ (again $j < k$), then R_{j+n} is any vector such that $dE(0, 0)\, R_{j+n} = \mu_j R_{j+n} + R_j$.

As the development continues, we shall discuss the problem in the case where all the a^2+4v_ib are positive, and we shall indicate the slight modifications in the alternate cases.

Now $\mu_i<0$ for $i\leqslant n$, and from (2.5), $\mu_{i+n}<0$ for $i<k$, while $\mu_{i+n}\geqslant 0$ for $i\geqslant k$. Furthermore, (3.2) implies that for small ρ, $s>(\lambda_k+\lambda_{k-1})/2$ and for these ρ,

$$\mu_i<\beta=[-a+\sqrt{a^2+2b(\lambda_{k-1}-\lambda_k)}]/2b<0, \qquad \text{if} \quad i<n+k. \quad (3.7)$$

Also

$$\mu_i>\alpha=[-a+\sqrt{a^2+4b(\lambda_{k+1}-\lambda_k)}]/2b>0, \qquad \text{if} \quad i>n+k, \quad (3.8)$$

for all ρ. (If $a^2+4bv_j<0$, we get similar estimates for the real parts of the eigenvalues; these will suffice for our proof.) Using (3.2), we have $2b\mu_{n+k}=-a+\sqrt{a^2-2b\rho+O_2(\rho)}$, so a Taylor expansion about zero gives

$$\mu_{n+k}(0)=-\rho/2a+O_2(\rho). \quad (3.9)$$

We shall also need a similar estimate on $\mu_{n+k}(V_k(\rho))$. To this end, we recall from [16] that $l_k\, d^2F(r_k,r_k)=\nabla_k\cdot r_k=1$; see (2.2). Then using (3.1), we have, at $\rho=0$,

$$\frac{d}{d\rho}V_F(\rho)=\nabla\lambda_k\frac{dV_k(0)}{d\rho}=\nabla\lambda_k\cdot r_k=1.$$

Thus $\lambda_k(V_k(\rho))=\lambda_k(0)+\rho+O_2(\rho)$, so that $v_k(V_k(\rho))=\lambda_k(V_k(\rho))-s=\frac{1}{2}\rho+O_2(\rho)$, from (3.2). Hence, we have

$$\mu_{n+k}(V_k(\rho))=\rho/2a+O_2(\rho). \quad (3.10)$$

We define an inner product on $\mathbb{R}^{2n}$ based on the R_j's. Thus, if $p=\sum p_iR_i$, $q=\sum q_iR_i$, then

$$(p,q)=\sum p_iq_i. \quad (3.11)$$

(If $a^2+4bv_j=0$, the $p_{n+j}q_{n+j}$ term in (3.11) has the factor μ_j^{-2}.)

In what follows, we shall use μ, R and L in place of μ_{n+k}, R_{n+k} and L_{n+k}, respectively, and consider these only at $u=0$. We let

$$\mathscr{R}_-=[R_1,\ldots,R_{n+k-1}], \qquad \mathscr{R}_+=[R_{n+k+1},\ldots,R_{2n}], \qquad R=[R_{n+k}],$$

where the brackets denote the space spanned by these vectors. We write any $\underline{X}$ in $\mathbb{R}^{2n}$ in the obvious way as $\underline{X}=\underline{X}_++\underline{X}_-+\underline{\underline{X}}$, and if $T=dE(0,0)$, we can write $T = T_+ + T_- + \tilde{T}$. With this decomposition, together with Taylor's formula, we can write Eqs. (3.2) as

$$\begin{aligned} \bar{\underline{X}}'_- &= T_- \bar{\underline{X}}_- + O_2(\bar{\underline{X}}), \\ \bar{\underline{X}}'_+ &= T_+ \bar{\underline{X}}_+ + O_2(\bar{\underline{X}}), \\ \tilde{\bar{\underline{X}}}' &= \mu \tilde{\bar{\underline{X}}} + \tfrac{1}{2} L \cdot d^2 T(\bar{\underline{X}}^2) + O_3(\bar{\underline{X}}). \end{aligned} \tag{3.12}$$

We make the change of variable $\rho Y = \bar{\underline{X}}$ and extend our decompositions in the obvious way; e.g., $Y'_- = T_- Y_- + \rho O_2(Y)$.

We define three sets B, B_1, and B_2 by

$$B = \{ Y \in \mathbb{R}^{2n} \colon |Y_-| \leqslant \varepsilon, |Y_+| \leqslant \varepsilon; -\tfrac{1}{2} \leqslant \tilde{Y} \leqslant \tfrac{3}{2} \},$$
$$B_1 = \{ Y \in B \colon \tilde{Y} \leqslant \tfrac{1}{2} \}, \qquad B_2 = \{ Y \in B \colon \tilde{Y} \geqslant \tfrac{1}{2} \}.$$

We now have our basic lemma.

LEMMA 3.1. *For sufficiently small ρ, and sufficiently small ε (ε independent of ρ), the sets B, B_1, and B_2 are isolating blocks.*

Proof. On the set $|Y_-| = \varepsilon$,

$$(|Y_-|^2)' = 2(Y_-, T_- Y_-) + \rho O(1) \leqslant 2\beta\varepsilon^2 + \rho O(1) < 0$$

for sufficiently small ρ, by (3.7). (If $a^2 + 4bv_j < 0$, the same estimate holds, where β is an upper bound for the real parts of the eigenvalues associated with $\mathscr{R}_-$. If $a_2 + 4bv_j = 0$, the estimate is valid with β replaced by $\beta/2$. The proof of this reduces to the fact that if $m < 0$, and $(x, y) \cdot (u, v)$ is defined to be $xu + 4v/m^2$, then $(mx + y, my) \cdot (x, y) \leqslant m |(x, y)|^2/2$.) Hence $\{|Y_-| = \varepsilon\} \subset b^+$; similarly, $\{|Y_+| = \varepsilon\} \subset b^-$.

As is to be expected, the flow on the other three boundaries is more delicate. Let $d = L \cdot R$; then (3.5) and (3.6) imply that $d = b(\mu_{n+k} - \mu_k) = \sqrt{a^2 + 4v_k b}$. Thus for sufficiently small ρ,

$$a \leqslant d \leqslant 5a/4, \tag{3.13}$$

since $v_k = \lambda_k - s = O(\rho)$, by (3.2).

From (3.12) and (2.2) we have

$$\begin{aligned} \tilde{Y}' &= \mu \tilde{Y} + (2d)^{-1} \rho L \cdot d^2 T(Y^2) + \rho^2 O_3(Y) \\ &= \mu \tilde{Y} + (2d)^{-1} \rho [\tilde{Y}^2 L \cdot d^2 T(R^2) + \varepsilon O(1)] + \rho^2 O_3(Y) \\ &= \mu \tilde{Y} + (2d)^{-1} \rho [\tilde{Y}^2 + \varepsilon O(1)] + \rho^2 O_3(Y), \end{aligned}$$

so that (3.9) gives

$$\tilde{Y}' = \rho[\tilde{Y}^2/d - \tilde{Y}/a + \varepsilon O(1)]/2 + O_2(\rho). \tag{3.14}$$

Using (3.13), it is easy to check that for ε sufficiently small (independent of ρ), and ρ sufficiently small, that $\tilde{Y}'$ will be negative for $\tilde{Y} = -\frac{1}{2}$, positive for $\tilde{Y} = \frac{1}{2}$ and negative for $\tilde{Y} = \frac{3}{2}$. This completes the proof.

We next compute the indices of these blocks. We use the notation Σ^k to enote a pointed k-sphere, and $\bar{O}$ to denote the trivial homotopy class, a (pointed) point. Finally we let S, S_1 and S_2 denote the maximal invariant sets contained in the isolating blocks B, B_1 and B_2 respectively.

LEMMA 3.2. *$h(S_1) = \Sigma^{n+k-1}$, $h(S_2) = \Sigma^{n+k}$ and $h(S) = \bar{O}$.*

Proof. From the last lemma, $b_1^+ = B_1 \cap \{|Y_-| = \varepsilon\}$, $b_2^+ = B_2 \cap [\{|Y_-| = \varepsilon\} \cup \{|\tilde{Y} - 1| = \frac{1}{2}\}]$ and $b^+ = \partial B \cap (b_1^+ \cup b_2^+) = \{|Y_-| = \varepsilon\} \cup \{|\tilde{Y}| = \frac{3}{2}\}$. Hence $h(S_1) = [B_1/b_1^+] = \Sigma^{n+k-1}$ and $h(S_2) = [B_2/b_2^+] = \Sigma^{n+k}$. To see that B/b^+ has trivial homotopy type, we observe that (B, b^+) is homeomorphic to $D^{n-k} \times (D^{n+k}, J)$, where J is the boundary of the $(n+k)$-cube, minus one of its faces. Thus D^{n+k} can be deformed into J along the lines of the standard projection from an exterior point.

COROLLARY 3.3. *There is an orbit of* (3.3) *which has its ω-limit set in B_2 and its α-limit set in B_1.*

This is an immediate consequence of Theorem 2.2 and Lemma 3.2.

In order to apply Corollary 3.3, we must show that Eqs. (3.3) are gradient-like in each B_i, and that $(0, 0)$ is in B_1, while $(u_r, 0)$ is in B_2.[4]

We consider first the set B_1 and show that for small ρ, the function

$$Q(Y) = |Y_-|^2 - |Y_+|^2 + \tilde{Y}^2$$

increases on (non-constant) orbits of (3.11) in B_1. We have

$$\begin{aligned} Q' = {} & 2(T_- Y_-, Y_-) + 2(T_+ Y_+, Y_+) + \rho(|Y_-| - |Y_+|)\, Q_2(Y) \\ & + \tilde{Y}^2[\tilde{Y}/\gamma - 1/a + \varepsilon(O(1))]\, \rho + O_2(\rho). \end{aligned}$$

If $M = \min(\alpha, -\beta)$, then the sum of the first two terms is at least $2M(|Y_+|^2 + |Y_-|^2)$. Since $\gamma \geqslant a$, $(\tilde{Y}/\gamma - 1/a) \leqslant -1/2a$ on B_1, so for small ε, independent of ρ, $(\tilde{Y}/\gamma - 1/a + \varepsilon(O(1))) \leqslant -1/4a$ and thus the fourth term

[4] From our remarks following (3.2), we may assume that these are the only ret points in B.

will be greater than $|\rho|\,\tilde{Y}^2/4a$. Next, there is a K such that the third term exceeds $\rho K\,|Y|^2\,(|Y_+|+|Y_-|)$. We thus have

$$\begin{aligned}4aQ' &\geqslant 8aM(|Y_+|^2+|Y_-|^2)+|\rho|\,\tilde{Y}^2+4a\rho K\,|Y|^2\,(|Y_+|+|Y_-|)+O_2(\rho)\\ &=8aM(|Y_+|^2+|Y_-|^2)+4a\rho K(|Y_+|^2+|Y_-|^2)(|Y_+|+|Y_-|)\\ &\quad+\rho\tilde{Y}^2+4a\rho K\tilde{Y}^2(|Y_+|+|Y_-|)+O_2(\rho)\\ &=|\rho|\,\{4a(|Y_+|^2+|Y_-|^2)[2M/|\rho|-K(|Y_+|+|Y_-|)]\\ &\quad+\tilde{Y}^2[1-4aK(|Y_+|+|Y_-|)]+O(\rho)\}\\ &\geqslant|\rho|\,\{8a(|Y_+|^2+|Y_-|^2)(M/|\rho|-\varepsilon K)+\tilde{Y}^2(1-8aK\varepsilon)+O(\rho)\}.\end{aligned}$$

Now choose ε such that $8aK\varepsilon<1$; then choose $|\rho|$ such that $\varepsilon K\,|\rho|<M$ and smaller if necessary to make the entire expression in brackets positive. With these choices we obtain $Q'>0$ in B_1. (The modifications needed when a^2+4bv_j is not positive are just as before.)

We now turn our attention to the set B_2. We make the change of variable $\rho Y+u_r=\underline{\bar{X}}$ and observe that $\rho Y'=E(\underline{\bar{X}})$, and

$$Y'=dE_{u_r}Y+\tfrac{1}{2}\rho\,d^2E_{u_r}Y^2+\rho^2O_3(Y).$$

We let

$$P(Y)=|Y_+|^2-|Y_-|^2-\tilde{Y}^2,$$

and attempt to show that P increases on (non-constant) orbits in B_2.

We have

$$\begin{aligned}P'(Y)&\geqslant\alpha_\rho\,|Y_+|^2-\beta_\rho\,|Y_-|^2-\rho\tilde{Y}^2(1/a+z/\gamma_\rho)\\ &\quad+\rho O_2(Y)(|Y_+|+|Y_-|)+O_3(\rho).\end{aligned}\tag{3.15}$$

The terms α_ρ, β_ρ, γ_ρ reflect the shift of the center for the Taylor expansion of E from the origin $V_k(0)$ to $V_k(\rho)$. These changes have the effect of multiplying the entire inequality by a term $1+O(\rho)$, which is to say, no effect at all, since we are considering only small ρ.

Let $\theta=(u_r/\rho,0)$; then from (3.1) and our change of scale, $\theta=(r_k+O(\rho),0)$. Hence

$$\begin{aligned}\tilde{\theta}&=\theta\cdot L/\gamma=(r_k+O(\rho))\cdot(-b\mu_k l_k)/\gamma\\ &=(1+O(\rho))(a+\sqrt{a^2+4bv_k})/2\sqrt{a^2+4bv_k}\\ &=(1+O(\rho))(\tfrac{1}{2}+O(\rho)+\tfrac{1}{2})=(1+O(\rho)).\end{aligned}$$

Also, $\theta_++\theta_-=\theta-\tilde{\theta}R=(r_k+O(\rho),0)-(1+O(\rho))(r_k,b\mu r_k)=O(\rho)$. Thus $\theta\in B_2$ for ρ sufficiently small.

Now since $\tilde{\theta} = 1 + O(\rho)$, we can achieve both $\tilde{Y} \leqslant 0.6$ and $|Y_{\pm}| \leqslant 2\varepsilon$ by choosing ρ sufficiently small. Then choosing ε small (independent of ρ), we have, from (3.15), $P'(Y) > 0$ on $B_2 \backslash \theta$. Using Theorem 2.2, we see that we have proved the following theorem.

THEOREM 3.4. *Let a be a fixed positive number; then (aI, bI) is an admissible pair for the hyperbolic genunity nonlinear system of conservation laws* (2.1), *for all sufficiently weak shocks.*

As in [5], we have the following corollary.

COROLLARY 3.5. *Under the above assumptions on system* (2.1), *if the pair (A, B) is sufficiently close to a pair (aI, bI), where $a > 0$, then (A, B) is admissible for all sufficiently weak shocks.*

4. SYSTEMS OF TWO CONSERVATION LAWS AND SHOCKS OF ANY STRENGTH

We consider the class of systems in two dependent variables studied in [16]:

$$u_t + f(u, v)_x = 0, \qquad v_t + g(u, v)_x = 0. \tag{4.1}$$

We require that $f_v < 0$, and $g_u < 0$; this makes system (4.1) hyperbolic; i.e., if $F = (f, g)$, then dF has real and distinct eigenvalues $\lambda_1(u, v) < \lambda_2(u, v)$. If l_k and r_k denote, respectively, the corresponding left and right eigenvectors, $k = 1, 2$, then we also require that $l_k\, d^2k(r_j, r_j) > 0$, $j, k = 1, 2$, where we have normalized the eigenvectors by $l_k r_k > 0$, $k = 1, 2$. As was shown in [15], these systems admit shocks of arbitrary strength. If $U = (u, v)$, then we consider the associated system containing both dissipation and dispersion terms:

$$U_t + F(U)_x = \varepsilon A U_{xx} + \varepsilon^2 B U_{xxx}. \tag{4.2}$$

We shall only consider matrices A and B of the form

$$A = \begin{pmatrix} a & c \\ c & d \end{pmatrix}, \qquad B = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix},$$

where A is positive definite, and $b_i \neq 0$, $i = 1, 2$. We call attention to the fact that our proof of admissibility will go through with only minor modifications if some of the $b_i = 0$; for brevity we shall omit this discussion.

Given any shock-wave solution of (4.1), our goal is to find conditions on (A, B) which render this pair admissible. These conditions will depend only on the strength of the shock, and will therefore be valid for all shocks whose strengths do not exceed a given value. Finally, we shall give the details only for 1-shocks (i.e., shocks corresponding to the eigenvalue λ_1; see [16]), and we shall leave to the reader the slight modifications necessary for 2-shocks. Recall from (2.5) that for a 1-shock of speed s we have

$$s < \lambda_1(u_l, v_l), \qquad \lambda_1(u_r, v_r) < s < \lambda_2(u_r, v_r). \tag{4.3}$$

From the results of [16], if a shock curve breaks with a given quadrant, it will remain in that quadrant. Using (3.1), we see that the k-shock curve has r_k as its tangent vector at the origin. The eigenvalues of dF are

$$\begin{aligned}\lambda_1 &= \tfrac{1}{2}(f_u + g_v - \sqrt{(f_u - g_v)^2 + 4f_v g_u}) \\ &< \tfrac{1}{2}(f_u + g_v + \sqrt{(f_u - g_v)^2 + 4f_v g_u}) = \lambda_2.\end{aligned}$$

These correspond to right eigenvectors $(1, a_i)$, where $a_i = (\lambda_i - f_u)/f_v$, and $a_1 > 0 > a_2$. Thus the 1-shock curve breaks into quadrant I or III and the 2-shock curve is in quadrant II or IV.

As in Section 3, we consider the progressive wave sstem associated with (4.2):

$$\begin{aligned} u' &= w, \\ v' &= z, \\ b_1 w' &= f - su - aw - cz, \\ b_2 z' &= g - sv - cw - dz. \end{aligned} \tag{4.4}$$

Here s is the shock speed associated with the given shock wave solution of (4.1):

$$\begin{aligned}(u, v)(x, t) &= (0, 0), \qquad x < st, \\ &= (u_r, v_r), \qquad x > st.\end{aligned}$$

It satisfies the jump condition $s(u_r, v_r) = F(u_r, v_r)$. In [3] it was shown that the only zeros of $F(u, v) - s(u, v)$ are the origin and the point $(u_r, v_r) = \bar{Q}$. It follows that the only rest points of (4.4) are the origin O and the point $Q = (u_r, v_r, 0, 0)$. We seek an orbit that flows from the origin to Q.

The curves $f - su = 0$ and $g - sv = 0$ intersect precisely at the points O and $\bar{Q}$. The possibility exists that one or more of these curves might have asymptotes, as in Fig. 1, but, as the only non-notational modification this would entail would be to close the gap with a horizontal line segment, we assume that the picture is as in Fig. 2 (so the shock curve lies in the fourth quadrant), and we let S denote the compact region bounded by the two curves.

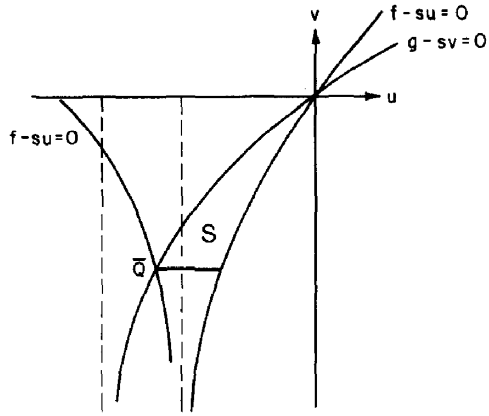

FIGURE 1

It is an easy matter to check that if the region S is as in Fig. 2, then the curve $g - sv = 0$ must be the "top" boundary of S. Indeed, since $s < \lambda_1(0,0)$ (by (4.3)), we have

$$s - f_u < 0, \qquad \text{and} \qquad s - g_v < 0, \qquad \text{at } (0,0). \tag{4.5}$$

Again from (4.3) we have, at $(0,0)$, $0 < (s - \lambda_1)(s - \lambda_2) = (s - f_u)(s - g_v) - f_v g_u$, so that the slopes of the curves $f - su = 0$ and $g - su = 0$ satisfy the inequalities $(s - f_u)/f_v > g_u/(s - g_v) > 0$. The curves are thus orientied as claimed. Note too that $f_v < 0$ implies that along $f - su = 0$, v is a function of u; $v = v(u)$. Similarly $u = u(v)$ along $g - sv = 0$. Finally, note that $f - su$ and $g - sv$ are each negative in the interior of S, and, in fact, also on the boundary curve opposite each one's zero curve (except at 0 and $\bar{Q}$).

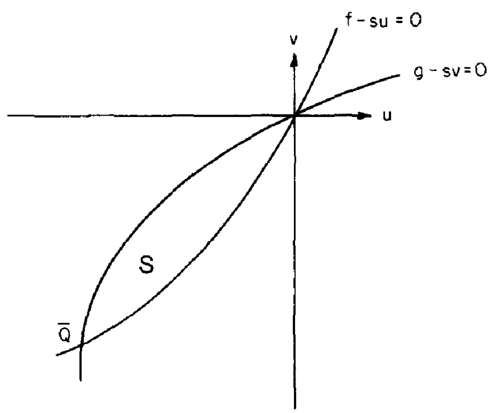

FIGURE 2

We introduce positive constants Δ, M and Γ by

$$\Delta = ad - c^2, \qquad \Delta M = 1, \qquad \Gamma = a + d + 2|c|.$$

Let $\varepsilon > 0$, and let S_ε denote the compact region determined by the curves $f - su = \varepsilon$ and $g - sv = \varepsilon$; obviously $S \subset S_\varepsilon$. Let

$$T_\varepsilon = \{(u, v, w, z): (u, v) \in S_\varepsilon, |f - su - aw - cz| \leqslant \delta_\varepsilon(u, v), \\ \text{and } |g - sv - cw - dz| \leqslant \delta_\varepsilon(u, v)\}, \quad (4.7)$$

where $\delta_\varepsilon(u, v)$ will be determined later.

We shall show below that T_ε is an isolating neighborhood for Eqs. (4.4) if ε is small enough. In preparation for this, we must examine the nature of the rest points O and Q. We shall show that the rest points are always hyperbolic; this follows from the positive definiteness of A. To see this, write system (4.4) in the form

$$\begin{aligned} BU' &= W, \\ W' &= F - sU - AB^{-1}W, \end{aligned} \qquad (4.8)$$

where $U = (u, v)$, $W = (w, z)$, and, as above, $F = (f, g)$. If we linearize this system at a rest point, then the eigenvalue equation takes the form

$$\begin{bmatrix} 0 & B^{-1} \\ dF - sI & -AB^{-1} \end{bmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \lambda \begin{pmatrix} \xi \\ \eta \end{pmatrix}.$$

We thus get the equations

$$\begin{aligned} B^{-1}\eta &= \lambda\xi, \\ (dF - sI)\,\xi - AB^{-1}\eta &= \lambda\eta. \end{aligned}$$

Thus $\xi \neq 0$ and

$$[\lambda^2 B + \lambda A - (dF - sI)]\,\xi = 0.$$

If we take the inner product of this equation with ξ we get the equation

$$\langle B\xi, \xi\rangle \lambda^2 + \langle A\xi, \xi\rangle \lambda + \langle (sI - dF)\,\xi, \xi\rangle = 0.$$

Since $\langle A\xi, \xi\rangle > 0$, this equation has no purely imaginary eigenvalues. We thus have proved the following lemma.

LEMMA 4.1. *If A is positive definite, then the rest points of* (4.4) *are hyperbolic; i.e., the linearized matrix about a rest point has no purely imaginary eigenvalues.*

We shall now show that T_ε is an isolating neighborhood for Eqs. (4.4) if ε is sufficiently small. Let $\bar{Q} = (u_r, v_r)$, and for $\varepsilon > 0$, let (see Fig. 3)

$$h_\varepsilon = \{(u, v)\colon u + v \leqslant \bar{u} + \bar{v} + \varepsilon\} \cap S_\varepsilon, \qquad h_0 = \bar{Q},$$
$$k_\varepsilon = \{(u, v)\colon u + v \geqslant +\varepsilon\} \cap S_\varepsilon, \qquad k_0 = 0.$$

Furthermore, let

$$H_\varepsilon = \{(u, v, w, z)\colon (u, v) \in h_\varepsilon, |f - su - aw - cz| \leqslant \delta'_\varepsilon(u, v), \text{ and } |g - sv - cw - dz| \leqslant \delta'_\varepsilon(u, v)\},$$
$$K_\varepsilon = \{(u, v, w, z)\colon (u, v) \in k_\varepsilon, |f - su - aw - cz| \leqslant \delta'_\varepsilon(u, v), \text{ and } |g - sv - cw - dz| \leqslant \delta'_\varepsilon(u, v)\},$$

where δ'_ε will be chosen later. Observe that since O and Q are hyperbolic rest points, we have the following lemma.

LEMMA 4.2. *For sufficiently small* $\varepsilon > 0$, *and sufficiently small* $\delta'_\varepsilon = \sup\{\delta'_\varepsilon(u, v)\colon (u, v) \in h_\varepsilon \cup k_\varepsilon\} > 0$, H_ε *and* K_ε *are isolating neighborhoods for* (4.4), *and isolate* Q *and* O, *respectively.*

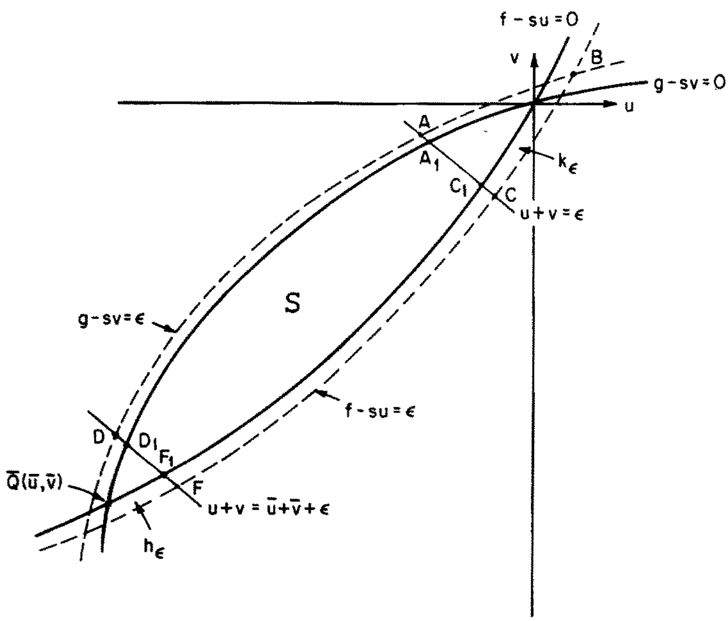

FIGURE 3

We consider now the set $T_\varepsilon \backslash (H_\varepsilon \cup K_\varepsilon)$, and prove that by suitably restricting the entries in A and B, the set $J_\varepsilon = \mathrm{cl}[T_\varepsilon \backslash (H_\varepsilon \cup K_\varepsilon)]$ is an isolating neighborhood for small $\varepsilon > 0$; that is, we can find $\delta_\varepsilon(u, v) > 0$ so that J_ε is an isolating neighborhood for small $\varepsilon > 0$.

In what follows, θ and ϕ will denote numbers between ± 1.

We must show that the flow through each boundary point of J_ε leaves J_ε in one or the other time direction.

Let

$$\alpha = \min\{\inf_S |f_v|, \inf_S |g_u|\} > 0, \tag{4.12}$$

and

$$\beta = \max\{\sup_S |\nabla f|, \sup_S |\nabla g|\} + |s|. \tag{4.13}$$

Note that in the set T_ε, both of the following inequalities are valid:

$$\begin{aligned} |c(f - su) - a(g - sv) + z\Delta| &\leqslant (a + |c|)\, \delta_\varepsilon, \\ |c(g - sv) - d(f - su) + w\Delta| &\leqslant (d + |c|)\, \delta_\varepsilon. \end{aligned} \tag{4.14}$$

Let $\tilde{P}$ be on the arc $D_1 A_1$ (cf. Fig. 3), and assume that

$$0 < d\alpha - |c|\, \beta \equiv m_1. \tag{4.15}$$

Then using (4.14), (4.15) and the fact that $f - su < 0$ at $\tilde{P}$,

$$\begin{aligned} (g - sv)' &= (g_v - s)\, z + g_u w \\ &= (g_v - s)\, M[a(g - sv) - c(f - su) + \psi(a + |c|)\, \delta_\varepsilon] \\ &\quad + g_u M[d(f - su) - c(g - sv) + \theta(d + |c|)\, \delta_\varepsilon] \\ &= (f - su)\, M[dg_u - c(g_v - s)] + M[g_u \theta(d + |c|)\delta_\varepsilon] \\ &\quad + M(g_v - s)\, \psi(a + |c|)\, \delta_\varepsilon \\ &> (f - su)\, M[dg_u - c(g_v - s)] - M\Gamma\beta\delta_\varepsilon \\ &= M(su - f)(-dg_u + cg_v - cs) - M\Gamma\beta\delta_\varepsilon \\ &\geqslant M[(su - f)(d\alpha - |c|\, \beta) - \Gamma\beta\delta_\varepsilon] \\ &= M[(su - f)\, m_1 - \Gamma\beta\delta_\varepsilon] \geqslant 0, \end{aligned}$$

if

$$\delta_\varepsilon \leqslant (su - f)\, m_1 / \Gamma\beta, \tag{4.16}$$

on the (closed) arc $D_1 A_1$. Thus, $(g - sv)' > 0$ at these points. Hence, if ε is small, then $(g-sv)'>0$ on the open arc DA so that these points will be strict-exit points on ∂T_ε.

Similarly, if

$$0<a\alpha-|c|\beta\equiv m_2, \tag{4.17}$$

then $(f-su)'>0$ provided that

$$\delta_\varepsilon \leqslant (sv-g)\, m_2/\Gamma\beta \tag{4.18}$$

on the (closed) arc F_1C_1. Hence $(f-su)'>0$ at all points on the open arc FC for small ε; thus these points will also be strict-exit points on ∂T_ε.

Let $j_\varepsilon = \mathrm{cl}[S_\varepsilon \backslash (h_\varepsilon \cup k_\varepsilon)]$; $j_0=S$. We define $\delta_\varepsilon(u,v)$ on j_ε, by

$$\delta_\varepsilon = [m_1(su-f)+m_2(sv-g)]/\beta\Gamma. \tag{4.19}$$

It is clear that for sufficiently small $\varepsilon>0$, δ_ε is always positive in j_ε, and that (4.16) and (4.18) are satisfied.

We have determined the nature of the flow on the "sides" of J_ε, and we now consider the "top" and "bottom" of J_ε. These are actually four hypersurfaces, one for each of the expressions defining T_ε, to equal $\pm\delta_\varepsilon$.

We begin with the hypersurface

$$f-su-aw-cz=\delta_\varepsilon, \tag{4.20}$$

and compute

$$\begin{aligned} H &= [f-su-aw-cz-\delta_\varepsilon]' \\ &= [f-su-aw-cz-(m_1(su-f)+m_2(s_v-g))/\beta\Gamma]' \\ &= [K_1(f-su)+G_2(g-sv)-aw-cz]', \end{aligned}$$

where $K_i=1+m_i/\beta\Gamma$ and $G_i=m_i/\beta\Gamma$. Thus

$$H=w[K_1(f_u-s)]+G_2\,g_u+z[K_1f_v+G_2(g_v-s)]+a\delta_\varepsilon/b_1-\theta c\delta_\varepsilon/b_2. \tag{4.21}$$

We intend to make the next-to-last term dominant, and this leads us to require

$$a/|b_1|-|c|/|b_2|>0. \tag{4.22}$$

Observe that on hypersurface (4.20), (4.14) implies that

$$w=M[d(f-su)-c(g-sv)-(d-\phi c)\,\delta_\varepsilon],$$

and

$$z=M[a(g-sv)-c(f-su)-(a\phi-c)\,\delta_\varepsilon],$$

so that since $\delta_\varepsilon = G_1(su-f)+G_2(sv-g)$ (by (4.19),

$$w = M\{(f-su)[d+(d-\phi c)\,G_1] + (g-sv)[-c+(d-\phi c)\,G_2]\},$$

and

$$z = M\{(f-su)[-c+(a\phi-c)\,G_1] + (g-sv)[a+(a\phi-c)\,G_2]\}.$$

Substituting these quantities in (4.21), we get

$$\begin{aligned} H = {} & (su-f)\{M[(\phi c-d)\,G_1-d][K_1(f_u-s)+G_2\,g_u] \\ & + M[c-(c-a\phi)\,G_1][K_1 f_v + G_2(g_v-s)] + G_1(a/b_1-\theta c/b_2)\} \\ & + (sv-g)\{M[c-(\phi c-d)\,G_2][K_1(f_u-s)+G_2\,g_u] \\ & + M[(c-a\phi)\,G_2-a][K_1 f_v+G_2(g_v-s)] + G_2(a/b_1-\theta c/b_2)\} \\ \equiv {} & (su-f)\,H_1 + (sv-g)\,H_2. \end{aligned}$$

Now one verifies immediately that $|G_i| \leqslant 1$, and $|K_i| \leqslant 2$, so that on $j_0 \equiv S$

$$\begin{aligned} H_1 &> (a/|b_1| - |c|/|b_2|)\,G_1 - 6M\beta\Gamma, \\ H_2 &> (a/|b_1| - |c|/|b_2|)\,G_2 - 6M\beta\Gamma. \end{aligned} \tag{4.23}$$

Hence, if on S both inequalities

$$\begin{aligned} (a/|b_1| - |c|/|b_2|)\,G_1 - 6M\beta\Gamma &> 0, \\ (a/|b_1| - |c|/|b_2|)\,G_2 - 6M\beta\Gamma &> 0 \end{aligned} \tag{4.24}$$

hold, then H is negative on hypersurface (4.20) provided that $(u,v) \in S\backslash\{0, Q\}$. Thus, for sufficiently small $\varepsilon > 0$, $H < 0$ on $\mathrm{cl}[S_\varepsilon\backslash(h_\varepsilon \cup k_\varepsilon)]$. Thus, orbits starting on hypersurface (4.20) which are in $\mathrm{cl}[T_\varepsilon\backslash(H_\varepsilon \cup K_\varepsilon)]$ move into the region $T_\varepsilon\backslash(H_\varepsilon \cup K_\varepsilon)$; i.e., they are entrance points.

Similarly, those boundary points in $\mathrm{cl}[T_\varepsilon\backslash(H_\varepsilon \cup K_\varepsilon)]$ which lie on the hypersurface $f-su-aw-cz=-\delta_\varepsilon$ are exit points, provided that (4.24) holds. Proceeding in the same way for the hypersurface $g-sv-cw-dz=\pm\delta$, we find that if

$$d/|b_2| - |c|/|b_1| > 0, \tag{4.25}$$

and if on S

$$(d/|b_2| - |c|/|b_1|)\,G_i - 6M\beta\Gamma > 0, \qquad i=1,2, \tag{4.26}$$

then those boundary points on $\mathrm{cl}[T_\varepsilon\backslash(H_\varepsilon \cup K_\varepsilon)]$ are also either strict-exit or strict-entrance points.

We now define δ'_ε in the sets K_ε and H_ε by

$$\delta'_\varepsilon(u, v) = \delta_\varepsilon(u_1, \varepsilon - u_1), \qquad \text{if} \quad (u, v) \in K_\varepsilon,$$

where (u, v) and $(u_1, \varepsilon - u_1)$ lie on the same level set of $f - su$.

δ'_ε is defined in h_ε in a similar way; see Fig. 4. Thus, under the assumption that O and Q are hyperbolic rest points of (4.4), we have proved the first part of the following lemma.

LEMMA 4.3. *For sufficiently small* $\varepsilon > 0$, *the set* T_ε *is an isolating neighborhood for* (4.4) *provided that* δ_ε *is sufficiently small and* (4.15), (4.17), *and* (4.24), (4.26) *hold on* S. *The Conley index of* T_ε *is* $\bar{O}$, *the trivial homotopy class.*

To see that T_ε has trivial homotopy type, we can deform B to a scalar matrix $\bar{b}I$ through positive diagonal matrices, and we can also deform A to a scalar matrix $\bar{a}I$ through positive definite matrices, keeping intact all of the estimates in the statement of the theorem, as well as keeping O and Q hyperbolic rest points. The continuation theorem for the Conley index implies that T_ε is an isolating neighborhood of the new flow, and has the same index. Using the continuation theorem again, we can deform s and (u_r, v_r) along the "shock-curve" (see [15]) so that the resulting shock is as weak as we choose; the Conley index for this continued isolating neighborhood is still invariant. By choosing the shock sufficiently weak and taking ε and δ_ε sufficiently small, we can conclude from Lemma 3.2 that this new isolating neighborhood has index $\bar{O}$; the same is then true for T_ε, and the proof of the lemma is complete.

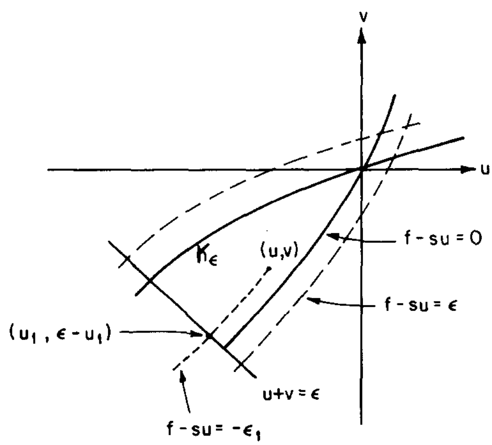

FIGURE 4

From Theorem 2.1 we see that there is an orbit of (4.4) which stays in T_ε and is different from O and Q. That this orbit connects O and Q will follow from the next lemma.

LEMMA 4.4. *If* (4.15) *and* (4.17) *hold, then the equations are gradient-like in* J_ε *provided that both* ε *and* δ_ε *are sufficiently small. In fact, the function* $(u+v)$ *decreases along orbits in* J_ε.

Proof. From (4.14), we have, along J_0,

$$\begin{aligned}(u+v') &= w+z\\ &= M\{(a-c)(g-sv)+(d-c)(f-su)+\delta_\varepsilon[\phi(a+|c|)+\theta(d+|c|)]\}.\end{aligned}$$

Since $\beta>\alpha$, (4.15) and (4.17) show $d>c$ and $a>c$. Therefore, if $\Gamma\delta_\varepsilon<(a-c)(sv-g)+(d-c)(su-f)$, then $(u+v)'<0$ in J_0; hence $(u+v)'<0$ in J_ε for small ε.

We can now state our main theorem (cf. Lemma 4.1).

THEOREM 4.6. *If* (4.15) *and* (4.17) *hold, and if* (4.24) *and* (4.26) *hold in* S*, then the pair* (A, B) *is admissible.*

Proof. As we noted above, for small ε, there is a complete orbit γ of (4.4) in T_ε, different from O and Q. This orbit canot lie completely in H_ε or K_ε, by Lemma 4.2. Thus γ meets $T_\varepsilon\backslash(H_\varepsilon\cup K_\varepsilon)$. Using Lemma 4.4 we see that the α- and ω-limit sets of γ are in H_ε and K_ε, respectively. Thus γ runs from O to Q, and (A, B) is admissible.

Of course, if the assumptions of Theorem 4.5 hold, then any pair $(\tilde{A}, \tilde{B})$ sufficiently close to the admissible pair (A, B) will also be admissible. Note too that (4.24) and (4.26) will hold if $|b_1|$ and $|b_2|$ are small relative to the other quantities. Thus Theorem 4.6 shows that the pair (A, B) is admissible for a given shock wave if B is "small" relative to A.

Observe that the conditions in the theorem simplify considerably if A is diagonal. In this case (4.5) and (4.17) always hold, and we have the following corollary.

COROLLARY 4.7. *If* A *and* B *are diagonal matrices, with* A *positive, then* (A, B) *is admissible provided that both* (4.24) *and* (4.26) *hold in* S*. In this case* (4.24) *and* (4.26) *can be written as*

$$\frac{6\beta^2}{\alpha}\leqslant\frac{ad}{(a+d)^2}\min(a, d)\min\left(\frac{a}{|b_1|}, \frac{d}{|b_2|}\right). \tag{4.27}$$

We shall discuss our conditions in the next section.

5. Examples and Discussion

We shall first make some general remarks and then give some examples.

Conditions (4.15) and (4.17) place restrictions on how far A is from a diagonal matrix. As we have already observed, they are always true for diagonal A. Conditions (4.22) and (4.25) must necessarily hold if (4.24) and (4.26) hold. They imply the validity of the inequalities

$$\frac{a}{|c|} > \left|\frac{b_1}{b_2}\right| > \frac{|c|}{d},$$

and are thus a restriction on how far B is from a scalar matrix, based on how far A is from a diagonal matrix. On the other hand, if A and B are scalar matrices, $A = aI$, $B = bI$, with $a, b > 0$, then (4.27) cannot hold if $a^2 < 6b$.

To understand these conditions somewhat better, we look at a special example. Consider the "p-system"

$$u_t - v_x = 0, \qquad v_t + p(u)_x = 0,$$

where $p' < 0$, and $p'' > 0$. This system is hyperbolic and satisfies all of the conditions discussed at the beginning of Section 1. We take A to be diagonal and consider again the case of 1-shocks. In view of Corollary 4.6, we need only check (4.27). This becomes

$$\frac{6[\max(1, \sup|p'(u)|) + |s|]^2}{\min(1, \inf|p'(u)|)} \leqslant \frac{ad}{(a+d)^2}\min(a, d)\min\left(\frac{a}{|b_1|}, \frac{d}{|b_2|}\right). \tag{5.1}$$

Taking $p(u) = u^{-\gamma}$ (isentropic gas dynamics), and $\bar{u} \geqslant u \geqslant \underline{u}$, then the left-hand side of (5.1) is at most

$$\frac{6[\max(1, \gamma/\underline{u}^{1+\gamma}) + \sqrt{\gamma}/\underline{u}^{(1+\gamma)/2}]^2}{\min(1, \gamma/\bar{u}^{1+\gamma})},$$

since $-s^2 = p'(\xi)$ for some ξ between $\underline{u}$ and $\bar{u}$.

As mentioned in the Introduction, we shall now give an example of a gradient system (4.1) in which every pair (A, B) is admissible. Here A and B are each positive definite matrices, and B is symmetric.

Let $G(u, v) = u^3 - uv$, and let $F(u, v) = \nabla G = (3u^2 - v, -u)$. Then

$$dF = \begin{bmatrix} 6u & -1 \\ -1 & 0 \end{bmatrix},$$

and the eigenvalues of dF are λ_1, λ_2, where

$$\lambda_1 = 3u - \sqrt{9u^2+1} < 0 < 3u + \sqrt{9u^2+1} = \lambda_2.$$

The corresponding right and left eigenvectors are

$$\begin{aligned} r_1 &= (-\lambda_1, 1)^t, \qquad & r_2 &= (\lambda_2, -1)^t, \\ l_1 &= (1, \lambda_2), & l_2 &= (1, \lambda_1), \end{aligned}$$

and since $d^2F[(x, y)]^2 = (6x^2, 0)^t$, the conditions $l_i r_i > 0$ and $l_i\, d^2F(r_j)^2 > 0$ are satisfied.

We consider only 2-shocks. Let $(0, 0)$, and (u_r, v_r) be a given shock-wave solution of speed s, $s > 0$, of the hyperbolic system $(u, v)_t + F(u, v)_x = 0$. Since r_2 points into the second quadrant, the "shock curve" lies in the fourth quadrant (see [16]).

Taking A to be positive definite and B symmetric and invertible, Eqs. (2.8) become

$$\begin{aligned} \begin{pmatrix} u \\ v \end{pmatrix}' &= C \begin{pmatrix} w \\ z \end{pmatrix}, \qquad C = B^{-1}, \\ \begin{pmatrix} w \\ z \end{pmatrix}' &= \begin{pmatrix} 3u^2 - v \\ -u \end{pmatrix} - AC \begin{pmatrix} w \\ z \end{pmatrix} - s \begin{pmatrix} u \\ v \end{pmatrix}. \end{aligned} \tag{5.2}$$

Let $U = (u, v)^t$, $W = (w, z)^t$ and $U_r = (u_r, v_r)^t$. If

$$K(U) = G(U) - \tfrac{1}{2} s \langle U, U \rangle,$$

and

$$H(U, W) = \tfrac{1}{2} \langle CW, W \rangle - K(U),$$

then

$$H' = -\langle CW, ACW \rangle < 0 \qquad \text{if} \quad W \neq 0.$$

Hence (5.2) is gradient-like along the set $W \neq 0$. (If $W = 0$, then $W' \neq 0$ except at the rest points of (5.2).) Note that $H(0, 0) = 0$.

We shall find a ball containing the origin in $\mathbb{R}^4$ which contains an orbit in the unstable manifold of the origin. This orbit then has non-empty ω-limit set and must tend to $(U_r, 0)$ as $t \to +\infty$. This will prove the admissibility of (A, B).

We begin by studying $K^{-1}(0)$. If $K = 0$, then $u^3 - uv - s(u^2 + v^2)/2 = 0$, so that

$$sv = -u \pm \sqrt{(1 - s^2)\, u^2 + 2su^3}. \tag{5.3}$$

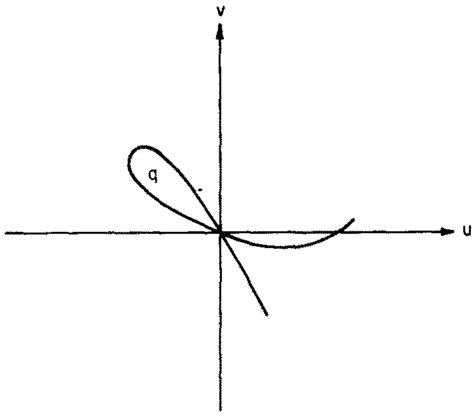

FIGURE 5

The jump-conditions yield

$$su_r = 3u_r^2 - v_r, \qquad sv_r = -u_r,$$

so that $s^2 = (1 + 3v_r)^{-1} < 1$, since $v_r > 0$. Thus from (5.3) we see that $K^{-1}(0)$ has two solutions if $u > 0$; if $u = 0$ then $v = 0$; if $(s^2 - 1)/2s < u < 0$, $K^{-1}(0)$ has two solutions; if $u = (s^2 - 1)/2s$, $v = (1 - s^2)/2s$. Finally, $K^{-1}(0) \cap \{u < (s^2 - 1)/2s\}$ is empty. Observe too that (5.3) shows $v > 0$ if $u < 0$. It follows that $K^{-1}(0)$ is as in Fig. 5, where we have divided $u < 0$ into the compact region q and its complement. Observe that $K > 0$ in q.

If $K(U) \leqslant 0$, then $H(U, W) > 0$ unless $W = 0$. Thus no orbit in the unstable manifold[5] of the origin can pass through a point (U, W) with $K(U) \leqslant 0$. It follows that there is an orbit in the unstable manifold of the origin which enters the 4-dimensional infinite cylinder Q whose cross-section is q. Once entering Q, it can never escape since $H \geqslant 0$ on

$$\partial Q = \{(U, W) : K(U) = 0, u \leqslant 0\}.$$

If $m = \sup\{K(U) : U \in q\}$, and r is chosen so large that $\langle CW, W \rangle > 2m$ if $|W| > r$, then this orbit can never escape the ball of radius r centered at the origin, since H is positive on the complement of this ball in Q. The ω-limit set of this orbit must also lie in the ball. But since the flow is gradient-like, the ω-limit set is $(U_r, 0)$, the only available critical point.

[5] The determinant of the linearization of (5.2) about 0 is $(s^2 - 1) \det C$, so that this unstable manifold is non-void.

Note that if $B = 0$, then $(A, 0)$ is admissible for every positive definite A. To see this, merely write the equations in the form $AU = F(U) - sU$, and note that $-K(U)$ decreases on orbits. Thus similar arguments as before go through.

We end this paper by noting that we have no example where $(A, 0)$ is admissible but (A, B) is not (or vice-versa). In fact, the technique of showing inadmissibility (for 1-shocks, if $n = 2$) is to show that 0 is an attractor; see [3].

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REFERENCES

1. E. B. BYKHOVSKII, Non-admissible viscosity matrices for equations of isothermal gas motion, *Dokl. Akad. Nauk SSSR* **146** (1963), 751–753; English transl., *Soviet Math. Dokl.* **3** (1962), 1372–1374.
2. C. CONLEY, "Isolated Invariant Sets and the Morse Index," CBMS Regional Conference Series, Vol. 38, Amer. Math. Soc., Providence, R. I., 1978.
3. C. CONLEY AND J. SMOLLER, Viscosity matrices for two-dimensional non-linear hyperbolic systems, *Comm. Pure Appl. Math.* **23** (1970), 867–884.
4. C. CONLEY AND J. SMOLLER, Viscosity matrices for two dimensional nonlinear hyperbolic systems, II, *Amer. J. Math.* **44** (1972), 631–650.
5. C. CONLEY AND J. SMOLLER, Shock waves as limits of progressive wave solutions of higher order equations, *Comm. Pure Appl. Math.* **24** (1971), 459–472.
6. C. CONLEY AND J. SMOLLER, Shock waves as limits of progressive wave solutions of higher order equations, II, *Comm. Pure Appl. Math.* **25** (1972), 133–146.
7. C. CONLEY AND J. SMOLLER, Topological methods in the theory of shock waves, *in* "Partial Differential Equations," pp. 293–302, Proceedings of Symposia in Pure Mathematics Vol. 23, Amer. Math. Soc., Providence, R. I., 1973.
8. C. CONLEY AND J. SMOLLER, On the structure of magnetohydrodynamic shock waves, *Comm. Pure Appl. Math.* **27** (1974), 367–375.
9. J. CONLIN, A theorem in ordinary differential equations with an application at hyperbolic conservation laws, *Adv. in Math.* **35** (1980), 1–18.
10. P. GORDON, Paths connecting elementary critical points of dynamical systems, *SIAM J. Appl. Math.* **26** (1977), 35.
11. P. LAX, Hyperbolic systems of conservation laws, II, *Comm. Pure Appl. Math.* **10** (1957), 537–566.
12. A. MAJDA AND J. RALSTON, Discrete shock profiles for systems of conservation laws, *Comm. Pure Appl. Math.* **32** (1979), 445–482.
13. M. MOCK, A topological degree for orbits connecting critical points of autonomous systems, *J. Diffential Equations* **38** (1980), 176–191.
14. R. SHAPIRO, "Shock Waves as Limits of Progressive Wave Solutions of Higher Order Equations," Ph. D. thesis, University of Michigan, 1973.

15. J. SMOLLER, On the solution of the Riemann problem with general step data for an extended class of hyperbolic systems, *Mich. Math. J.* **16** (1969), 201–210.
16. J. SMOLLER AND J. JOHNSON, Global solutions for an extended class of hyperbolic systems, *Arch. Rational Mech. Anal.* **32** (1969), 169–189.
17. J. SMOLLER AND M. TAYLOR, Wave front sets and the viscosity method, *Bull. Amer. Math. Soc.* **79** (1973), 431–436.